Dynamic, Stochastic Models for Congestion Pricing and Congestion Securities

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16. Abstract

This research considers congestion pricing under demand uncertainty. In particular, a robust optimization (RO) approach is applied to optimal congestion pricing problems under user equilibrium. A mathematical model is developed and an analysis performed to consider robust, dynamic user equilibrium, optimal tolls based on the second-best problem known as the dynamic optimal toll problem with equilibrium constraints, or DOTPEC. Finally, numerical experiments and qualitative analyses are conducted to investigate the performance and robustness of the solutions obtained.

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DYNAMIC, STOCHASTIC MODELS FOR CONGESTION PRICING, AND CONGESTION SECURITIES

FINAL REPORT

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By

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The Thomas D. Larson Transportation Institute

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1. Introduction

Congestion pricing is widely studied as an efficient method for managing transportation congestion externalities by inducing travel behavior that minimizes social cost. Typically, congestion pricing models assume that demand is known in advance and deterministic values of demands are used to find optimal tolls. However, system performance can be negatively impacted when inelastic, deterministic demands are employed, especially when demands depart significantly from their expected nominal values (Waller et al. 2001, Gardner et al. 2008). Also, precise travel demands are virtually impossible to obtain, due to specification errors and imperfect data that plague real-world forecasting. Accordingly, in this report, we consider robust congestion pricing problems in the presence of transportation demand uncertainty.

Since the initial exploration of road pricing by Pigou (1920), both the theoretical and applied literatures on congestion pricing has grown rapidly. The congestion pricing problem may be classified using four criteria: (1) first-best or second-best pricing, (2) static or dynamic traffic assignment, (3) homogeneous or heterogeneous users, and (4) deterministic or stochastic parameters. The literature review presented herein focuses on second-best congestion pricing problems, particularly in dynamic traffic assignment with homogeneous users, instead of providing a comprehensive survey on congestion pricing problems. A more comprehensive review is provided by Yang and Huang (2005).

Since the first-best pricing problem calculates tolls based on the difference between social and private marginal cost over all links in a network, it is thought by many not to be applicable to real-world traffic networks. Therefore, second-best pricing, wherein a subset of arcs can be tolled, is gathering increasing attention from researchers and practitioners (Lindsey and Verhoef 2001, Lawphongpanich and Hearn 2004). In the case of a dynamic transportation network, Henderson (1974) explained the importance of departure time decisions and showed the influence of time in varying congestion tolls using the single bottleneck model by Vickrey (1969). Subsequently, congestion pricing for the bottleneck model has been investigated by various researchers (Arnott et al. 1990, Arnott and Kraus 1998, Yang and Huang 1998, Braid 1996, De Palma and Lindsey 2000). However, these works have limited their attention to very simple networks and have not stressed computability. In the context of general networks, Carey and Srinivasen (1993) provided analytical approximate expressions for congestion tolls using the Kuhn-Tucker optimality condition. Wie and Tobin (1998) formulated a convex optimal control problem for first-best dynamic marginal tolls. A simulation-based analysis to determine the impact of six types of link tolling schemes was conducted by De Palma et al. (2005). Lin et al. (2010) proposed a heuristic combining dual variable approximation techniques using a linear programming model based on the cell transmission model. There are several papers considering a bi-level or MPEC formulation of second-best pricing. Viti et al. (2003) proposed a framework for the joint choice of route and departure times in light of tolls; therein departure time and route choice are modeled sequentially and a simple grid search approach is used to find optimal, uniform tolls. By contrast, Joksimovic et al. (2005), model departure time and route choice simultaneously, while also using a simple grid search to find optimal, uniform as well as time-varying tolls. Wie (2007) assumed triangular shaped, multi-step congestion tolls to maximize consumer surplus and proposed the Hooke-Jeeves algorithm to compute them.

In the area of congestion pricing under uncertainty, Gardner et al. (2008) proposed a stochastic mathematical programming model with equilibrium constraints to determine robust first-best tolls in the presence of uncertain demand. The objective of that effort was to minimize a weighted sum of expected total travel time and standard deviation for a finite number of pre-determined demand scenarios. Nagae and Akamatsu (2006) formulated a stochastic singular control problem for second-best toll pricing. In their report, toll price was selected from a set of tolls to maximize expected net profit value.

Our research differs from the aforementioned work in some important ways. Our robust optimization (RO) approach means that we will not assume the availability of a probability distribution for the underlying uncertain data. Moreover, our RO approach guarantees feasibility through the use of prescribed uncertainty sets (e.g., see Ben-Tal and Nemirovski 1999 and Bertsimas and Sim 2004).

Recently, an RO approach was employed by Ban et al. (2009) for robust road pricing corresponding to multiple traffic assignment solutions with fixed demand. Lou et al. (2010) also studied robust congestion pricing; their aim was to minimize total system travel time in light of a distribution of all possible boundedly rational user equilibriums.

The main focus of this report is the formulation and solution of robust congestion pricing problems in which only a subset of the links in a transportation network can be tolled. We apply a robust optimization approach to the dynamic user equilibrium optimal toll problem with demand uncertainty. In describing uncertain demand, it is assumed that uncertain demands are drawn from a predefined box uncertainty set. As we shall show, this perspective is acceptable when a decision maker wants to determine tolls for a specific transportation network component (such as a tunnel, bridge or highway) without knowing exact travel demand distributions.

One challenging aspect of robust congestion pricing is the user equilibrium condition. Under user equilibrium, ignoring demand uncertainty for the moment, the nominal problem can be modeled with a bi-level formulation as a mathematical program with equilibrium constraints (MPEC) that expresses user equilibrium condition as a variational inequality (VI) appearing as a family of constraints. We consider the dynamic optimal toll problem with equilibrium constraints (DOTPEC) with demand uncertainty. We note that there are several types of toll collecting policies in the literature (e.g., uniform toll in Viti et al. (2003), time-varying toll selection with pre-determined price levels in Nagae and Akamatsu (2006), the triangular shape in Wie (2007), etc.). For simplicity, we assume that the toll shapes are determined in advance through an external selection process that specifies triangular-shaped tolls. Due to certain properties of triangular-shaped tolls that are discussed subsequently, it is only necessary to decide the value of the maximum toll for each tolled arc in order to determine the toll trajectory within any pre-determined time interval. This toll-setting mechanism is coupled to the deterministic DOTPEC problem to formulate a robust counterpart; we then apply the cutting plane algorithm in conjunction with a simulated annealing algorithm to set optimal dynamic user equilibrium tolls.

2. Robust Congestion Pricing for Dynamic Traffic Networks

In this section, as a foundation of a robust dynamic congestion pricing problem, we first introduce a deterministic DOTPEC problem which has been studied by Friesz et al. (2007). The key portion of the DOTPEC problem is the time-shifted DUE formulation in network loading part given in Friesz et al. (2001). As flow propagation constraints hold the time-shift in equation, it is difficult to handle them, especially in the computation perspective. However, Friesz et al. (2011) derive the DAE system that describes the network loading when the point queue model is invoked, and it may be efficiently and accurately approximated using a related system of ordinary differential equations by using the second-order Taylor expansion for flow propagation constraints. In the following sections, we briefly describe the network loading approach in Friesz et al. (2011) and introduce the deterministic DOTPEC problem as well as the robust counterpart. For discussion convenience, the notations used throughout these models are presented in Table 1.

Table 1. Notations

Symbol	Description
$i, j \in N$	Nodes in the network
$a \in A$	An arc in the network
$w \in W$	An origin-destination pair
$p \in P_w$	A path between OD pair
$\Delta = [\Delta_{ap}]$	The arc-path incidence matrix
$\Lambda = [\Lambda_{wp}]$	The OD pair-path incidence matrix
$Q = [Q_w]$	The vector of traffic demand
$h = [h_p]$	The vector of path flows
$f = [f_a]$	The vector of arc flows
$c_p(h) = [c_p(h)]$	The vector of path travel time function
$\theta(h, y) = [\theta_p(h, y)]$	The vector of path travel cost function with toll
$y = [y_a]$	The vector of congestion toll on arc a
$\delta = [\delta_a]$	The vector of defining tolled arc; $\delta_a = 1$ if arc a is tolled, otherwise 0

2.1 Dynamic Network Loading

The purpose of the dynamic network loading is to find arc flow and delay when travel demand and departure rates (path flows) are given. Dynamic network loading is closely related to the determination of path delays, since arc delay is the sum of the delays on paths using an arc. Mathematically, the volume on arc a is written as

$$x_a(t) = \sum_{p \in P} \delta_{ap} x_a^p(t) \quad \forall a \in A \tag{1}$$

where $x_a^p(t)$ denotes the volume on arc a associated with path p and

$$\delta_{ap} = \begin{cases} 1 & \text{if arc } a \text{ belongs to path } p \\ 0 & \text{otherwise.} \end{cases}.$$

To describe flow propagation, we make use of the simple deterministic arc delay model in Friesz et al. (1993). By the recursive relationship of exit time function, the exit time from arc for path $p \in P$ is defined as

$$\xi_{a_1}^p = t + D_{a_1}\left[x_{a_1}(t)\right] \quad \forall p \in P \tag{2}$$

$$\xi_{a_i}^p = \xi_{a_{i-1}}^p(t) + D_{a_i}\left[x_{a_i}\left(\xi_{a_{i-1}}^p(t)\right)\right] \quad \forall p \in P, i \in \left[2, m(p)\right] \tag{3}$$

where $D_{a_i}\left[x_{a_i}(t)\right]$ is the time required to travel on arc a_1 of every path $p = \{a_1, a_2, ..., a_{m(p)}\} \in P$. The exit time function allows us to derive the following flow propagation constraints (see details in Friesz et al. 2001)

$$g_{a_1}\left(t + D_{a_1}\left[x_{a_1}(t)\right]\right)\left(1 + D'_{a_1}\left[x_{a_1}(t)\right]\dot{x}_{a_1}\right) = h_p(t) \tag{4}$$

$$g_{a_i}^{p}\left(t+D_{a_i}\left[x_{a_i}(t)\right]\right)\left(1+D'_{a_i}\left[x_{a_i}(t)\right]\dot{x}_{a_i}(t)\right)=g_{a_{i-1}}^{p}(t) \quad \forall p\in P, i\in\left[2,m(p)\right] \tag{5}$$

where $h_p^k (= g_{a_0}^{p,k})$ and $g_{a_i}^{p}(t)$ are the departure rate and exit flow on arc a_i along path p at time t, respectively. A second order Taylor series approximation of equations 4 and 5 yields

$$g_{a_1}^{p}\left(t+D_{a_1}\left[x_{a_1}(t)\right]\right)\approx g_{a_1}^{p}(t)+\frac{dg_{a_1}^{p}(t)}{dt}D_{a_1}\left[x_{a_1}(t)\right]+\frac{d^2g_{a_1}^{p}(t)}{dt^2}\frac{\left(D_{a_1}\left[x_{a_1}(t)\right]\right)^2}{2} \quad \forall p\in P \tag{6}$$

$$g_{a_i}^{p}\left(t+D_{a_i}\left[x_{a_i}(t)\right]\right)\approx g_{a_i}^{p}(t)+\frac{dg_{a_i}^{p}(t)}{dt}D_{a_i}\left[x_{a_i}(t)\right]+\frac{d^2g_{a_i}^{p}(t)}{dt^2}\frac{\left(D_{a_i}\left[x_{a_i}(t)\right]\right)^2}{2} \quad \forall p\in P, i\in\left[2,m(p)\right] \tag{7}$$

Next, the total traversal time for path p may be expressed using equations 2 and 3:

$$D_p=\sum_{i=1}^{m(p)}\left[\xi_{a_i}^{p}(t)-\xi_{a_{i-1}}^{p}(t)\right]=\xi_{a_{m(p)}}^{p}(t)-t \quad \forall p\in P \tag{8}$$

We assume that the effective delay includes an arrival penalty operator F; thus, the effective delay operator is

$$c_p=D_p+F\left[t+D_p-T_A^{\tau}\right] \quad \forall p\in P \tag{9}$$

where T_A is the desired arrival time and $F\left[t+D_p-T_A^{\tau}\right]=0.5(t+D_p-T_A^{\tau})^2$. When the departure rates $\hat{h}(t)$ are known, the arc exit flows, volumes and delays can be obtained. Let us define the traffic volumes given $\hat{h}(t)$ as

$$x_a(t)=\sum_{p\in P}\delta_{ap}x_a^{p}(t) \quad \forall a\in A. \tag{10}$$

Using equation 2, arc exit time of the first arc on a path can be computed, and then the arc exit time function for the remaining arcs in the path are computed with equation 3. After getting the path exit time $\xi_{a_{m(p)}}^{p}(t)$, the effective path delay may be computed as

$$c_p\left(t,h(t)\right)=\xi_{a_{m(p)}}^{p}(t)-t+F\left[\xi_{a_{m(p)}}^{p}(t)-T_A\right]. \tag{11}$$

2.2 Dynamic User Equilibrium

We have studied the approximated network loading in the previous section, which allows us to solve the dynamic user equilibrium efficiently. Furthermore, it is certain that we have to consider the efficient toll which should exist in the form of effective path delay operator. Hearn and Yildirim (2002) studied the efficient toll in the static congestion pricing with the linear travelling cost for traffic flow. The objective of the efficient toll is to make the user equilibrium traffic flow equivalent to the system optimum by appropriate congestion pricing. To study the dynamic efficient toll problem, we introduce the notion of a tolled effective delay operator:

$$\theta_p\left(t,h(t),y(t)\right)=D_p+F\left[t+D_p-T_A\right]+y_p(t) \quad \forall p\in P \tag{12}$$

where $y_p(t)$ denotes the toll for path p. It is easy to observe that

$$\theta_p\left(t,h(t),y(t)\right)=c_p\left(t,h(t)\right)+y_p\left(t\right). \tag{13}$$

Therefore, a dynamic tolled user equilibrium must obey

$$\sum_{p\in P}\int_{t_0}^{t_f}\theta_p\left(t,h^*\left(t\right),y\left(t\right)\right)\left(h\left(t\right)-h^*\left(t\right)\right)dt\geq 0,\quad \forall h\in\Omega \tag{14}$$

where $\Omega=\left\{\sum_{p\in P}\int_{t_0}^{t_f}h_p\left(t\right)dt=Q_w,h_p\left(t\right)\geq 0,w\in W\right\}$.

Dynamic user equilibrium is the solution from variational inequality equations, equation 14. However, we may mention that equation 14 is equivalent to a differential variational inequality (DVI). This can be shown easily by noting that the flow conservation constraints can be restated as

$$\begin{aligned}&\frac{ds_w}{dt}=\sum_{p\in P_w}h_p\left(t\right),\quad w\in W\\&s_w(t_0)=0,\quad w\in W\\&s_w(t_f)=Q_w,\quad w\in W\end{aligned} \tag{15}$$

which is recognized as a two-point boundary value problem. In equation 15, h and s_w are departure rate and dummy variable for flow conservation constrains with respect to total demand Q_w, respectively. Therefore, the constraints (14) may be expressed as the following differential variational inequality

$$\sum_{p\in P}\int_{t_0}^{t_f}\theta_p\left(t,h^*\left(t\right),y\left(t\right)\right)\left(h\left(t\right)-h^*\left(t\right)\right)\geq 0,\quad \forall h\in\Omega \tag{16}$$

where $\Omega=\left\{h\geq 0;\frac{ds_w}{dt}=\sum_{p\in P_w}h_p\left(t\right),s_w(t_0)=0,s_w(t_f)=Q_w,w\in W\right\}$.

We note that a vector of departure rates (path flows) $h^*\in\Omega$ is a dynamic user equilibrium if and only if h^* solves differential variational inequality. However, it is quite complicated to solve the differential variational inequality due to the fact that the effective path delay operator $\theta_p\left(t,h^*\left(t\right),y\left(t\right)\right)$ is typically neither monotonic nor differentiable. In this study, according to Friesz et al. (2011), we adopt an iterative fixed-point method in Hilbert space for a fixed-point problem equivalent to the differential variational inequality.

2.3 Robust Dynamic Congestion Pricing Problem

The DOTPEC is a type of dynamic network design problem for which a central authority (upper level objective function) tries to minimize congestion in a transport network whose flow obeys a dynamic network user equilibrium by dynamically adjusting tolls. In order to consider the dynamic optimal toll problem, it is obvious that the dynamic tolled user equilibrium and the dynamic system optimum problem should be considered at the same time. Consequently, the deterministic DOTPEC problem has the form of a dynamic system optimum objective function with the dynamic tolled user equilibrium as constraints. As mentioned before, the DOTPEC is a type of MPEC. Now, we introduce the DOTPEC problem as following

$$
\begin{aligned}
&\min z = \sum_{p\in P}\int_{t_0}^{t_f} c_p\left(t, h(t)\right) h_p(t) dt \\
&s.t. \\
&\sum_{p\in P}\int_{t_0}^{t_f} \theta_p\left(t, h^*(t), y(t)\right)\left(h(t) - h^*(t)\right) dt \geq 0, \quad \forall h \in \Omega \\
&y_{LB} \leq y \leq y_{UB}
\end{aligned}
\tag{17}
$$

where $\Omega = \left\{ \sum_{p\in P}\int_{t_0}^{t_f} h_p(t) dt = Q_w, h_p(t) \geq 0, w \in W \right\}$.

Next, we assume that the uncertain demand belongs to box uncertainty sets as defined in equation 18. The box uncertainty set is used in RO when the support of uncertain data is known, which is a relatively mild assumption on uncertain data and easy to get.

$$Q_w \in U_{Q_w} = \left[\bar{Q}_w(1-\theta), \bar{Q}_w(1+\theta)\right] \tag{18}$$

The given formulation for a deterministic DOTPEC problem is also an MPEC problem, which is a class of non-convex optimization problem. After introducing uncertainty set U_Q, we may recognize that the number of constraints for DVI is infinite due to the infinite number of demand scenarios in U_Q. This fact compounds the difficulty of the MPEC problem.

2.4 Solution Algorithm

In this section, we adopt a heuristic algorithm to deal with general cases inspired by Yin and Lawphongpanich (2007). They proposed a cutting plane (or constraint accumulation) algorithm for a robust network design problem and provided the convergence of the optimal solution under some assumptions. In order to apply a cutting plane algorithm approach to solve the robust counterpart, we need to assume a finite number of candidate total demands from equation 18:

$$\hat{Q} = \left\{Q^1, Q^2, ..., Q^n\right\} \tag{19}$$

then the relaxed robust DOTPEC (R-RDOTPEC) can be written as

$$
\begin{aligned}
&\min_{h,y,z_1} z_1 \quad \text{(R -RDOTPEC)} \\
&\text{subject to} \\
&\sum_{p\in P}\int_{t_0}^{t_f} c_p\left(t, h^i(t)\right) h_p^i(t) dt \leq z_1 \quad \forall i = 1, ..., n \\
&\sum_{p\in P}\int_{t_0}^{t_f} \theta_p\left(t, h^{i*}(t), y(t)\right)\left(h^i(t) - h^{i*}(t)\right) dt \geq 0, \quad \forall h^i \in \Omega\left(Q^i\right), i = 1, ..., n \\
&y_{LB} \leq y \leq y_{UB}
\end{aligned}
\tag{20}
$$

where $\Omega = \left\{ h^i \geq 0; \frac{ds_w^i}{dt} = \sum_{p \in P_w} h_p^i(t), s_w^i(t_0) = 0, s_w^i(t_f) = Q_w^i, Q_w^i \in \hat{Q}_w, w \in W \right\}$.

The optimal solution y^* is a robust optimal solution given finite number of uncertain demand scenario $\hat{Q}_w$. Then, the WCD problem is solved to find the worst-case scenario given ($\hat{y} = y^*$) and update $\hat{Q}_w$.

$$
\begin{aligned}
&\max z_2 \quad (WCD) \\
&s.t. \\
&\sum_{p \in P} \int_{t_0}^{t_f} c_p\left(t, h^i(t)\right) h_p^i(t) dt \leq z_2 \\
&\sum_{p \in P} \int_{t_0}^{t_f} \theta_p\left(t, h^{i*}(t), \hat{y}(t)\right)\left(h^i(t) - h^{i*}(t)\right) dt \geq 0, \quad \forall h^i \in \Omega\left(Q^i\right), i = 1, \ldots, n
\end{aligned}
\tag{21}
$$

where $\Omega = \left\{ h^i \geq 0; \frac{ds_w^i}{dt} = \sum_{p \in P_w} h_p^i(t), s_w^i(t_0) = 0, s_w^i(t_f) = Q_w^i, Q_w^i \in U_{Q_w}, w \in W \right\}$.

For a given $\hat{y}$, the objective function of WCD is to find a demand scenario in U_Q whose dynamic user equilibrium flow yields the maximum total travel cost. If Q^* is the solution to the WCD problem and z_2^* with Q^* is less than or equal to z_1^*, then $\hat{y}$ is a robust optimal toll. On the contrary, if z_2^* with Q^* is larger than z_1^*, then an improved solution may be obtained by solving the relaxed R- RDOTPEC problem by adding the Q^* to the demand scenario set such as $\hat{Q} = \hat{Q} \cup \left\{Q^*\right\}$. Finally, we may show that the algorithm itself has the form given below:

<u>*Cutting Plane Algorithm*</u>

$\boxed{\text{Step 0}}$ Set $k = 1$ and determine initial demand scenario $\hat{Q}^1$.

$\boxed{\text{Step 1}}$ Solve R- RDOTPEC with finite number of demand scenarios $\hat{Q}$. Let $\left(z_1^k, y^k\right)$ be the objective value and the optimal congestion price.

$\boxed{\text{Step 2}}$ Solve WCD with given toll price y^k. Let $\left(z_2^k, d^k\right)$ be the objective value and worst-case demand.

$\boxed{\text{Step 3}}$ If $z_2^k \leq z_1^k$, stop and y^k is a robust congestion price vector. Otherwise set $\hat{Q}^{k+1} = \hat{Q}^k \cup \left\{d^k\right\}$ and $k = k + 1$. Go to Step 1.

We note that the cutting plane algorithm is a heuristic approach and it can be applied for general problems with both convex and nonconvex uncertainty sets. It may be worth explaining how to solve the R-RDOTPEC problem and the WCD problem. There are a few papers on dynamic transportation network problems (e.g., Friesz et al. 2007; Wie 2007). In this study, we extend a simulated annealing approach applied by Friesz et al. (1992) to solve R-RDOCTPEC and WCD. Kirkpatrick et al. (1983) proposed a simulated annealing approach to find out a relation to the mechanics of annealing solids. The concept of a simulated annealing is based on the atomic state in the system. For example, if the system's temperature is high, its atoms are in a highly disordered state. Producing a more ordered state of atoms requires reducing the energy of the system by lowering its temperature. Atoms will achieve an equilibrium state at any fixed temperature. Interestingly, the scheme employed to reduce temperature may be applied as the form given below:

Simulated Annealing Algorithm

$\boxed{\text{Step 0}}$ Determine an initial value T^k (temperature at stage k), Q^k (step size distribution), $y^{(m,k)}$, and M (the number of iteration at each temperature stage); set $k=1$ for temperature stage and $m=1$ where m is the iteration at each temperature stage.

$\boxed{\text{Step 1}}$ Solve the DVI problems and find objective value $\Psi^{(m,k)}\left(y^{(m,k)}\right)$ for given $\hat{Q}_w$ and $y^{(m,k)}$ where $y^{(m,k)}$ is the value of y on the m^{th} step at k$^{\text{th}}$ temperature stage; otherwise go to Step 6.

$\boxed{\text{Step 2}}$ If $m<M$, in order to determine a candidate optimal solution, the enhancement variables are randomly perturbed from their current values according to

$$y^{(m+1,k)} = y^{(m,k)} + \Delta y^{(m,k)} \tag{22}$$

where $\Delta y^{(m,k)} = Q^k u$, u is a random vector, and each u_i is randomly and independently chosen from the normalized interval $\left[-\sqrt{3},\sqrt{3}\right]$.

$\boxed{\text{Step 3}}$ Solve the variational inequality problem for given $y^{(m+1,k)}$

$\boxed{\text{Step 4}}$ If $\Psi^{(m+1,k)}\left(y^{(m+1,k)}\right)<\Psi^{(m,k)}\left(y^{(m,k)}\right), y^{(m,k)} \leftarrow y^{(m+1,k)}$ and $m=m+1$. Then go to Step 2.

Otherwise go to Step 5.

$\boxed{\text{Step 5}}$ Calculate the

$$P\left(\Delta\Psi_k\right) = \exp\left(-\frac{\Psi^{(m+1,k)} - \Psi^{(m,k)}}{k_B T}\right) \tag{23}$$

where k_B is the Boltzman constant and compare with a random number $R \in [0,1]$. If R is less than or equal to $P\left(\Delta\Psi_k\right)$, then the $y^{(m,k)} \leftarrow y^{(m+1,k)}$ and $m=m+1$. Then go to Step 2. Otherwise the $y^{(m,k)} \leftarrow y^{(m,k)}$ and $m=m+1$ and go to Step 2.

$\boxed{\text{Step 6}}$ Calculate A_i^k and S_{ij}^k

$$A_i^k = \frac{1}{M}\sum_{m=1}^{M} y_{a_i}^{(m,k)} \tag{24}$$

$$S_{ij}^k = \frac{1}{M}\sum_{m=1}^{M}\left(y_{a_i}^{(m,k)} - A_i^k\right)\left(y_{a_j}^{(m,k)} - A_j^k\right) \tag{25}$$

The covariance matrix s for the next temperature stage $k+1$ is chosen as follows

$$s^{k+1} = \frac{\chi_s}{\beta M} S^k$$

where χ_s is the growth factor, typically > 1.

$\boxed{\text{Step 7}}$ Obtain Q^{k+1} corresponding to any desired covariance matrix s

$$s^{k+1} = Q^{k+1} \cdot \left(Q^{k+1}\right)^T$$

Then $y^{(1,k)} \leftarrow y^{(M,k)}, Q^k \leftarrow Q^{k+1}, T^k = 0.8T^k$ and $k = k+1$ .Then go to Step 1.

3. Numerical Example

The purpose of the numerical experiments in this section is to illustrate the advantage of the RO approach for congestion pricing under demand uncertainty. We implemented an algorithm with MATLAB and GAMS and solved a dynamic congestion pricing problem. A 3-arc and 4-node network in Figure 1 is considered to illustrate a robust DOTPEC problem. The set of OD pairs are $W = \{(1,4),(2,4)\}$ and uncertain travel demand for each OD pair; that is $Q_{1,4}, Q_{2,4} \in [90,110]$. The commuting period is between t_0 =08:00AM and t_f =09:00AM. The desired arrival time is 08:30AM for OD pairs (1,4) and (2,4). The parameters used for the linear arc delay function $D_a = A_a + y_a + B_a x_a$ are given by $4 + 0.003 f_1$, $3 + y + 0.0025 f_1$ and $2 + 0.002 f_1$ from arc 1 to 3, respectively. We solved this problem using the cutting plane algorithm proposed in Section 2, which is solved in around 53 minutes. The congestion toll is charged from 08:15AM to 08:45AM and the maximum toll of the R-DOTPEC problem is 9.78 with total cost of 5740.4. This means that the total travel cost is at most 5740.4 with the robust toll price. In contrast, the maximum toll of the deterministic solution is 8.71 and the objective value is 5154.3. Figure 2 shows the optimal dynamic congestion price on arc a_2 for both the deterministic problem and the robust counterpart. The path flow and tolled travel cost for the deterministic problem are shown in Figures 3 and 4.

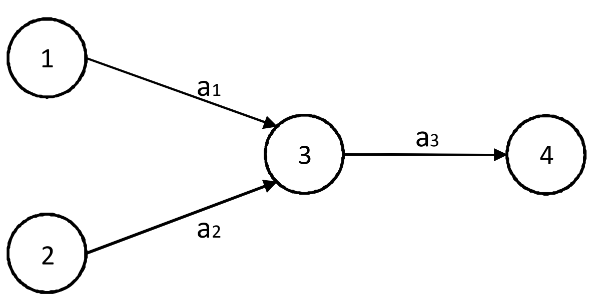

Figure 1. Three-arc, 4-node network

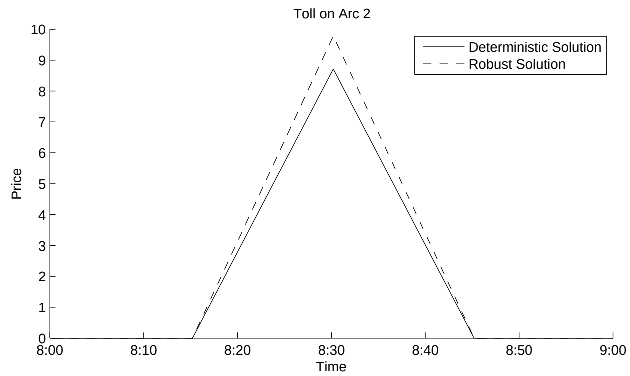

Figure 2. Toll price on arc a_2

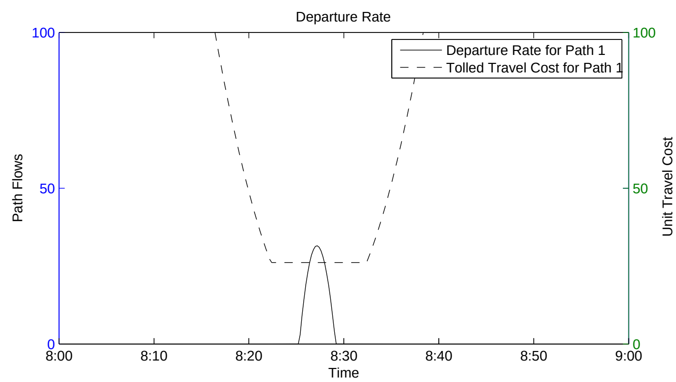

Figure 3. Departure rate and tolled unit travel cost for path 1

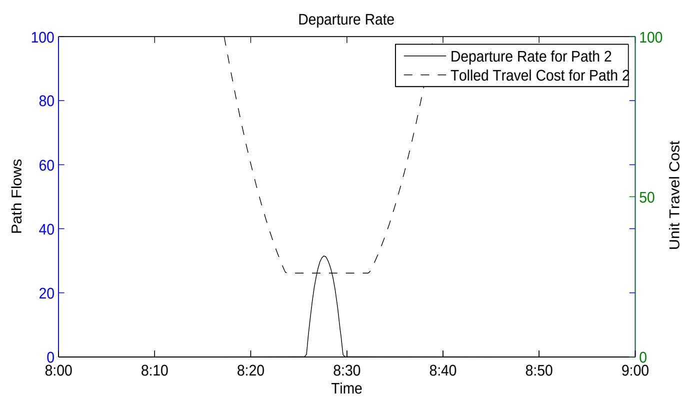

Figure 4. Departure rate and tolled unit travel cost for path 2

Next, we compare the robust solution with solutions from the deterministic problem. Simulation results are summarized in Table 2. Using the robust solution, the worst-case objective value among 100 random demands is 5740.4. The worst objective value is 5799.1 with the deterministic DOTPEC solution and sampling-based approach, respectively. We can see that the robust solution provides a guaranteed upper bound of objective value and performs well in terms of the worst-case solution, mean, and standard deviation, even though the improvement of the robust solution is not significant in this toy problem.

Table 2. Simulation results

	Nom	Rob	Imp
Worst Case	5799.1	5740.4	1%
Mean	5215.9	5182.9	1%
Standard Deviation	355.26	330.42	7%

4. Conclusion

In this study, we considered a robust optimization approach to user equilibrium optimal toll problems in dynamic transportation networks. A robust DOTPEC problem was solved iteratively using a cutting plane algorithm. During each iteration in the cutting plane algorithm, there were two problems to be solved: a relaxed robust DOTPEC and a worst-case demand problem. Furthermore, for each single problem, there were two sub-algorithms embedded: a fixed-point algorithm for solving variational inequality constraints, and a simulated annealing algorithm for solving a bi-level problem to get the toll price. For the numerical experiments, we used a 3-arc and 4-node network. As shown in the results, the worst-case result from the robust solution is better and the robust solution provides a more stable objective value than the nominal solution.

In future studies, we will consider a certain type of adjustable robust counterpart (ARC), in which the robust optimal solution is a function of uncertain demand to incorporate realized demand, which may gives us a less conservative solution for dynamic network problems. In particular, we can consider an affine adjustable robust counterpart to explore a tractable solution. In this study, we only considered fixed but uncertain demand data. We can extend our formulations to consider the elastic demand case and other types of uncertainty. For example, Ban et al. (2009) considered the uncertainty due to non-unique user equilibrium solution, and Lou et al. (2010) considered a solution set of boundedly rational user equilibrium. Also, there may be other sources of uncertainties, including preferred arrival time and parameters of penalty function. Finally, more efficient algorithms are highly desirable to solve practical large-scale problems.

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